Finding optimal wavelet bases of cascade processes

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Multiplicative cascade processes[1,?] are found in a wide range of different physical systems. In these systems, the energy or an analogous quantity is transferred from large to small scales through an independent scale-invariant factor, thus conferring a statistically self-similar behaviour to this quantity. Such a behaviour is usually referred to as *multifractality*, because it is related to the presence of a multifractal structure (a hierarchical combination of fractal sets), which is a very general case of self-similarity.[3] This way, the presence of multifractality allows to recognize the cascade process, either as a real mechanism or an effective one.[4] An example of multiplicative cascade process is the case of Fully Developed Turbulence (FDT),[5] where the cascade transfers the energy from large to small scales (where it is finally dissipated) giving rise to its multifractal structure, but such a behaviour is quite ubiquitous in nature and in fact has been observed in systems as diverse as stock market series,[3] natural images,[6] the heliospheric magnetic field, human gait, heartbeat dynamics,[7] network traffic, fractures, fire plumes, as well as many other complex systems.

While studying the cascade process is known to be a good strategy to obtain the global descriptors of a system (such as its multifractal characterization[4]), it is possible to also achieve a local dynamical description, thanks to the *optimal wavelet* of the system. The cascade process with the optimal wavelet describes a local effective dynamics that can be used in reconstruction of gaps or lost information, data compression and time-series forecast.

Wavelet transforms are integral transforms that allow to separate the details of a signal that are relevant at different scale levels. In other words, this means that wavelet transforms are precisely tuned to an adjustable scale and, for this reason, they are a powerful strategy to represent cascade processes. In addition, wavelets are Hilbert bases, i.e., the wavelet transform is invertible, so that the signal can be completely represented from the cascade process.[8] Given a signal $s(t)$, the wavelet transform at scale r is defined as $\alpha_r(t) = s \otimes \Psi_r$, where $\Psi_r(t) = \Psi(\frac{t}{r})$ and Ψ is a certain function called *wavelet*.

In a cascade process, the wavelet transform follows a multiplicative relation, i.e., two different scales r, L with $r < L$ are related through a multiplicative variable:

$$\alpha_r \doteq \eta_{r/L} \alpha_L \qquad (1)$$

where $\eta_{r/L}$ and α_L are mutually independent. Here, the symbol '$\doteq$' means that both sides equal in distribution, but not explicitly for each point t. The distribution of $\eta_{r/L}$ is usually used as a global descriptor of the system, as it determines the cascade process.

In practical cases, the cascade relation works with almost any wavelet Ψ. We propose the existence of an *optimal* wavelet for which the cascade relation is geometrically (and not only statistically) verified, meaning that the equality (1) holds at each point t of the signal.[10] This allows to retrieve the wavelet transform values at the smaller-scales from the one at the largest (whole-domain wide) scale, and through the $\eta_{r/L}$ distribution. Then, inverting the wavelet transform we can reconstruct the signal and infer missing points or even future ones. Existence of such a wavelet is not guaranteed, so its optimality must be checked *a posteriori*.

We will introduce a parameter to quantify the degree of optimality of a certain wavelet when faced a given dataset: $Q = \langle \frac{\alpha_r(t)}{\alpha_L(t)} \rangle$. We will prove that $Q = \sqrt{r/L}$ only with the optimal wavelet (biunivocally), while non-optimal wavelets always lead to higher values of Q, the farther from optimal the higher the value of Q. This parameter is rather simple and it is proven to be very robust and little data-demanding, while giving the same ranking in suboptimal wavelets as the mutual information $I\left(\frac{\alpha_r(t)}{\alpha_L(t)}, \alpha_L(t)\right)$ (which proves optimality if zero, but is little precise when faced to short datasets). Therefore, Q can be used as a cost function whose minimization leads to the optimal wavelet. Even further, as illustration of a promising application, we will find the optimal wavelet of a dataset formed by Spanish IBEX-35 time series and discuss how this wavelet can improve predictions based on the multifractal cascade.

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1 R. Benzi, G. Paladin, G. Parisi, A. Vulpiani, J. Phys. A **17**, 3521 (1984).
2 R. Benzi, L. Biferale, A. Crisanti, G. Paladin, M. Vergassola, A. Vulpiani, Physica D **65**, 352–358 (1993).
3 B. Mandelbrot, A. Fisher, and L. Calvet, *Cowles Foundation Discussion Paper No. 1164* (1997).
4 G. Parisi, U. Frisch, in *Proc. Intl. School of Physics E. Fermi*, (M. Ghil, R. Benzi and G. Parisi Eds., Amsterdam, 1985), pp 84–87
5 U. Frisch., *Turbulence.* Cambridge Univ. Press, Cambridge MA, 1995.
6 A. Turiel et al., *Phys. Rev. Lett.* **80**, 1098 (1998).
7 P. Ivanov et al., *Nature* **399**, 461 (1999).
8 I. Daubechies. *Ten lectures on wavelets.* CBMS-NSF Series in App. Math. Capital City Press, Montpelier, Vermont, 1992.
9 A. Chhabra et al., *Phys. Rev. A* **40**, 5284 (1989).
10 A. Turiel, N. Parga, Phys. Rev. Lett. **85**, 3325 (2000)