A WAVELET TECHNIQUE FOR THE REPRESENTATION OF MULTIFRACTAL DATA AS AN EXPLICIT TURBULENT CASCADE: APPLICATION TO TIME SERIES

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Multifractal systems are non-linear physical systems with some strong statistical invariances such as translational and scale invariance. The classical characterization of this kaleidoscopic scale invariance has been made through the Canonical Multifractal Formalism (CMF), which is a purely statistical approach. In CMF, the scale-invariant basic quantities of interest are the structure functions of order p (denoted by $S_p(r)$) of an appropriate intensive variable with variable scale scope r. For multifractal systems the structure functions follow power-laws: $S_p(r) \propto r^{\zeta_p}$ where the set of canonical exponents ζ_p characterize the scaling properties of the system.
It is possible to establish a link between statistics and geometry. Parisi and Frisch [1] proved that the statistical scale invariance could be recognized as the sign of an inner organization of the system in a hierarchy of fractal manifolds. The explicit decomposition of any given experimental signal into the associated critical manifolds requires the use of singularity analysis [2] and the framework provided by the Microcanonical Multifractal Formalism (μCMF) [3, 4]. The application of the μCMF to experimental data of different sources has allowed to widen and deepen the study of many different physical systems (natural images [2], heart-beat dynamics [5], econometric time-series [6], etc) and, in addition, the theory has gained conceptual depth. What, in the beginning, was a statistical description adapted to the peculiarities of Fully Developed Turbulence has proved to have deep roots in the study of non-linear, scale invariant systems.
In this context an open problem consists in rendering the connection among geometry and statistics explicit. One of the best known manifestations of CFM, and also the closest to an explicit geometrical realization of multifractal processes, concern the existence of multiplicative cascades. Let $\epsilon_r(\vec{x})$ be the appropriate, intensive variable of scope r which allows to unveil the multifractal structure of the signal (for instance in the case of turbulent fluids, $\epsilon_r(\vec{x})$ stands for the local energy dissipation on a ball of radius r centered around $\vec{x}$). Given two scales r, L such that $r < L$, the multiplicative cascade takes the form $\epsilon_r \doteq \eta_{r/L}\, \epsilon_L$, where the sign "$\doteq$" means that this equality cannot be considered to hold at any point $\vec{x}$, but only in a statistical sense (that is, both sides are statistically equally distributed). The variable $\eta_{r/L}$, which characterize the cascade process, is taken independent from ϵ_L and does not depend on each scale scope (r and L) separately but on their ratio,

r/L. If for any $0 < \kappa < 1$ the p-moments of η_κ scale as ζ_p, $\langle \eta_\kappa^p \rangle = \kappa^{\tau_p}$ it trivially follows that $\langle \epsilon_r^p \rangle \propto r^{\zeta_p}$ retrieving CMF.
In this work, we take these ideas as a starting point to propose a explicit representation of multifractal signals in an optimal wavelet basis for which the cascade process is geometrically (and not only statistically) implemented. Namely, if the wavelet projection of the signal s on the wavelet Ψ at the scale r and the point $\vec{x}$ is denoted by $T_\Psi s(\vec{x}, r)$, what we search is a wavelet Φ such that $T_\Phi s(\vec{x}, r) = \eta_{r/L}(\vec{x})\, T_\Phi s(\vec{x}, L)$, where now the equality holds for any point $\vec{x}$ and scales r, L. This requirement is too exigent and in principle the existence of such "optimal" Φ cannot be granted; however, if we restrict our choice of scales and position to those of a dyadic decomposition (for instance, in a Quadrature Mirror Filter representation), an explicit constructive formula for Φ, derived from a "learning" sample dataset, arises [7]. This formula proves unicity but not existence, so the "optimality" of the wavelet should be checked in each case. Interestingly enough, this formula makes a link with some constructive models of multifractals [8, 9].
We will concentrate our work on the study of multifractal time-series of different nature. Some prospects on future applications will be also presented.

References

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